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FIRST-COME-FIRST-SERVE SCHEDULING IS OFTEN OPTIMAL

PART I: SYMMETRY-CONVEXITY

by

JAMES R. JACKSON

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WESTERN MANAGEMENT SCIENCE INSTITUTE

University of California, Los Angeles

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ABSTRACT

This paper develops the theory of "symmetry-convex" functions, whose appearance in certain scheduling problems as delay-cost functions (of waiting times) guarantees that the "first-come-first-serve" schedules will be optimal. These are the symmetric functions satisfying a mild convexity condition (which appears to hold in most if not all practical scheduling situations). The conclusion of the present paper is that, for a rather special type of scheduling problem, the first-come-first-serve schedules are optimal, provided the waiting times of individual jobs affect delay cost symmetrically. Extensions of this conclusion (to be detailed in one or more subsequent papers) cover broad classes of scheduling problems of great practical significance, relating to both "first-come-first-serve" and "first-due-first-serve" scheduling.

1. INTRODUCTION

We consider the following scheduling problem.

(1.1) PROBLEM. Jobs $1,\ldots,n$ will arrive in a system, ready for processing, at epochs x_m, $m = 1,\ldots,n$. One job must be processed at each of the processing epochs y_m, $m = 1,\ldots,n$. A <u>schedule</u> is a permutation, say π, of the integers $1,\ldots,n$; with the interpretation that each processing epoch y_m is assigned to job $\pi(m)$; or, writing $\rho = \pi^{-1}$, that each job m is to be processed at epoch $y_{\rho(m)}$. Schedule π is <u>feasible</u> if it calls for processing each job no earlier than its arrival; i.e., if each $y_{\rho(m)} - x_m \geq 0$. Then $y_{\rho(m)} - x_m = w_m$ is the <u>waiting time</u> of job m. The <u>delay cost</u> depends upon the waiting times, being given by $\gamma(w_1,\ldots,w_n)$, where γ is a specified function. It is required to determine whether feasible schedules exist, and if so to choose one which minimizes delay cost.

A <u>first-come-first-serve</u> schedule is one under which two jobs which arrive in a given temporal order are always processed in the same order (or simultaneously); i.e., a schedule, π, such that if $x_i < x_j$, then $y_{\rho(i)} \leq y_{\rho(j)}$, where ρ is defined and interpreted as in the statement of Problem (1.1). It turns out that whenever feasible schedules exist, then the (easily constructed) first-come-first-serve schedules are feasible.

We shall take an indirect approach to the problem of minimizing delay cost. Specifically, we determine exactly what assumptions about the delay-cost function guarantee that the first-come-first-serve

schedules will minimize delay cost whenever feasible schedules exist.

Happily, rather general assumptions of practical applicability afford such a guarantee. The functions which satisfy these assumptions are called symmetry-convex functions in this paper, for reasons which will become clear. They are the symmetric functions which satisfy a certain mild convexity condition. It is plausible that practical delay-cost functions will always fulfill the convexity condition; and if this be accepted, we can conclude that the first- come-first-serve schedules are solutions to real-life instances of Problem (1.1) if and only if the individual jobs are symmetric with respect to their effects on delay cost. This last condition is often a matter of high principle in customer-service facilities, when managers accept egalitarian ideals or simply perceive the customers as a homogeneous population.

These conclusions are of interest in themselves, but much more so because they can be extended to a broad range of scheduling problems much more complex than Problem (1.1), and subject to realistic uncertainty. Suppose, for instance, that "future" arrival and processing epochs are not known in advance; and a dispatcher must decide which of several waiting jobs should be assigned a "present" processing opportunity. If the delay-cost function is symmetry-convex; he can confidently make his choice on the basis of the "first-come-first-serve rule" (assign a job which has been waiting longest), knowing that he will thus generate a first-come-first-serve schedule, and consequently minimize delay cost ... regardless of what the future holds. He need not even know which symmetry-convex function is operative, or how many jobs are yet to come; and he need not even make statistical assumptions beyond broadly accepting that his choices will not influence the (as yet unknown) configuration of future arrival and processing epochs in any useful way.

Such extensions comprehend a rather complete theory of scheduling to minimize delay cost in situations such as those represented by classical queueing models, provided that individual jobs are not statistically differentiated either as to effects on delay cost or as to processing time requirements. The circumstances covered include convincing representatives of a large proportion of customer-service operations, and of a substantial realm of manufacturing problems (e.g., repair of devices from a homogeneous population, when the required effort for a given job is <u>not</u> predictable before the job is irretrievably in process). The single dictum of the theory is the simple prescription, "use the first-come-first-serve rule." The detailed and formal development of these extensions is reserved, however, for another paper, for which the present paper lays groundwork.

<u>Outline</u>. In Section 2, the concept of "symmetry-convexity" is introduced for subsets of n-space, and this concept is developed in Sections 2 and 3. The symmetry-convex functions are defined in Section 4, and the main results concerning Problem (1.1) and relating to other anticipated applications are developed. Section 5 is a brief summary of results, especially as they are relevant to Problem (1.1). The Appendix rounds out portions of the mathematical theory of symmetry-convexity.

<u>Notation and terminology</u>. We use standard set-theoretic notations; such as "$\in$" for "belongs to" (a set), "$\supset$" and "$\subset$" for "contains" and "is contained in" (among sets), and "$\cap$" and "$\cup$" for (set) intersection and union. In particular, a notation of the form $\{$expression$|$list of conditions$\}$ signifies the set of elements representable by the "expression," and satisfying the "list of conditions"; and $\gamma : X \to Y$ denotes a completely general function whose values are defined for arguments in set X and are elements of set Y.

When no confusion can result, we may notationally identify a single element and the set consisting of that element.

Real n-space is denoted by R^n; and $R = R^1$. For $x \in R^n$, x_m and $(x)_m$ both denote the m-th component of x (the latter notation actually being used only when the element of R^n is denoted by a string of two or more symbols). For x and $y \in R^n$, the segment joining x and y is

$$[x,y] = \{\theta x + (1 - \theta)y \mid 0 \le \theta \le 1\};$$

where here and subsequently the arithmetic among elements of R and R^n is ordinary vector arithmetic.

The set of permutations of n objects is denoted by Π_n. For $x \in R^n$ and $\pi \in \Pi_n$, πx denotes the vector got from x by shifting the m-th component to position $\pi(m)$, $m = 1,\ldots,n$; so that $(\pi x)_m = x_{\pi^{-1}(m)}$ for each m. If also $\pi' \in \Pi_n$, then $\pi'\pi x = \pi'(\pi x)$. The notation and conventions are consistent with viewing the elements of R^n as column n-vectors and those of Π_n as nxn permutation matrices.

The symbol Σ_n denotes the set of $\sigma \in \Pi_n$ such that, for some i and j with $i \neq j$, $\sigma(i) = j$, $\sigma(j) = i$, and $\sigma(m) = m$ for $m \neq i, j$. Such a permutation is said to "exchange i and j." It is well-known (and readily verified) that every permutation can be expressed as a product of such "exchanges."

Set $X \subset R^n$ is <u>symmetric</u> if $\pi x \in X$ whenever $x \in X$ and $\pi \in \Pi_n$. Function $\gamma: X \to Y$, where $X \subset R^n$, is <u>symmetric</u> if X is symmetric and if $\gamma(x) = \gamma(\pi x)$ whenever $x \in X$ and $\pi \in \Pi_n$.

Set $X \subset R^n$ is <u>convex</u> if X is non-empty and if $[x,y] \subset X$ whenever x and $y \in X$. The <u>convex hull</u> of $X \subset R^n$ is the intersection of all convex sets containing X; or, equivalently (as is easily proved) the set,

$$\{\Sigma_{x \in X}\, \theta_x\, x \mid 0 \le \theta_x \le 1;\ \Sigma_{x \in X}\, \theta_x = 1\}.$$

Function $\gamma: X \to R$, where $X \subset R^n$, is convex if X is convex and if $\gamma(z) \leq \theta\gamma(x) + (1 - \theta)\gamma(y)$ whenever x and $y \in X$, $0 \leq \theta \leq 1$, and $z = \theta x + (1 - \theta)y$.

Function $\gamma: X \to R$, where $X \subset R^n$, is quasi-convex if X is convex and and if $\gamma(z) \leq \max \{\gamma(x); \gamma(y)\}$ whenever x and $y \in X$ and $z \in [x,y]$. It is well known (and easily proved) that every convex function is quasi-convex, but not conversely.

2. SYMMETRY-CONVEX SETS

As a basis for an appropriately general definition of the symmetry-convex functions, and also for analytic purposes, we first introduce the concept of symmetry-convexity for sets.

(2.1) DEFINITION. Set $X \subset R^n$ is symmetry-convex if whenever $x \in X$ and $\sigma \in \Sigma_n$, then segment $[x, \sigma x] \subset X$.

(2.2) DEFINITION. The symmetry-convex hull, $S(X)$, of set $X \subset R^n$ is the intersection of all symmetry-convex sets containing X.

Our first theorem states several elementary properties of symmetry-convexity.

(2.3) THEOREM. For $X \subset R^n$:

(2.3.1) $S(X) \supset X$.

(2.3.2) $S(S(X)) = S(X)$.

(2.3.3) X is symmetry-convex if and only if $S(X) = X$.

(2.3.4) $S(X)$ is symmetry-convex.

(2.3.5) If X is symmetry-convex, then X is symmetric.

(2.3.6) If $Y \subset X$, then $S(Y) \subset S(X)$.

(2.3.7) If X is a union or intersection of symmetry-convex sets, then X is symmetry-convex.

(2.3.8) $S(X) = \bigcup_{x \in X} S(x)$.

(2.3.9) X is symmetry-convex if and only if $X = \bigcup_{x \in X} S(x)$.

Remark. It is shown in the appendix that each set $S(x)$ is identical to the convex hull of the set, $\{\pi x \mid \pi \in \Pi_n\}$; whence (2.3.9) yields a rather

clear-cut characterization of the symmetry-convex sets in terms of ordinary convexity.

Proof of (2.3). Statement (2.3.1) is immediate from (2.2). Replacing X in (2.3.1) by $S(X)$, we have $S(S(X)) \supset S(X)$, and to prove (2.3.2) it remains to show that $S(S(X)) \subset S(X)$. Every set intersected to form $S(X)$ is a symmetry-convex set containing $S(X)$, and is hence among those intersected to form $S(S(X))$, whence the required relationship follows.

If X is symmetry-convex, then it is among the sets intersected to form $S(X)$, whence $S(X) \subset X$; which with (2.3.1) proves the necessity in (2.3.3). For $X \subset R^n$, if $x \in X$ and $\sigma \in \Sigma_n$, then x is in each of the sets intersected to form $S(X)$; whence, since these sets are symmetry-convex, each contains $[x, \sigma x]$; which implies $[x, \sigma x] \subset S(X)$. Consequently, if $S(X) = X$, then $[x, \sigma x] \subset X$; which proves the sufficiency in (2.3.3).

By (2.3.2), $S(X)$ satisfies the sufficient condition of (2.3.3) that it be symmetry-convex; which proves (2.3.4).

To prove (2.3.5), we assume X is symmetry-convex, and show that if $x \in X$ and $\pi \in \Pi_n$, then $\pi x \in X$. We can write $\pi = \sigma^K \ldots \sigma^1$, where each $\sigma^k \in \Sigma_n$. Let $y^0 = x$, and $y^k = \sigma^k y^{k-1}$ for $k = 1, \ldots, K$. Then $y^0 \in X$; and, by (2.1), $y^{k-1} \in X$ implies $y^k \in X$. This completes an induction showing that $y^K = \pi x \in X$, as required.

If $Y \subset X$; then, by (2.3.1) and (2.3.4), $S(X)$ is a symmetry-convex set containing Y, and hence among the sets intersected to form $S(Y)$; whence $S(Y) \subset S(X)$, which proves (2.3.6).

To prove (2.3.7), let $X = \bigcup_{a \in A} Z_a$ and $Y = \bigcap_{a \in A} Z_a$, where A is an arbitrary set and each Z_a is symmetry-convex. If $x \in X$ and $\sigma \in \Sigma_n$, then x is in some particular Z_a, so $[x, \sigma x] \subset Z_a \subset X$; which implies that X is symmetry-

convex, as required to prove the statement concerning unions. If $y \in Y$ and $\sigma \in \Sigma_n$, then y is in every Z_a, whence so is $[y, \sigma y]$, whence $[y, \sigma y] \subset \bigcap_{a \in A} Z_a = Y$; which implies that Y is symmetry-convex, and completes the proof.

To prove (2.3.8), write $U = \bigcup_{x \in X} S(x)$. By (2.3.6), $S(x) \subset S(X)$ for $x \in X$, whence $U \subset S(X)$. It remains to show that $U \supset S(X)$. By (2.3.4) and (2.3.7), U is symmetry-convex; and, by (2.3.1), $x \in S(x)$, whence $U \supset X$. Hence U is among the sets intersected to form $S(X)$, which implies the required relationship.

Finally, (2.3.9) is an immediate consequence of (2.3.3) and (2.3.8).

The next theorem provides a sufficient condition that a set be symmetry-convex. It is shown in the appendix that this condition is also necessary.

(2.4) THEOREM. Let $X \subset R^n$ be a union of symmetric sets whose intersections with the hyperplanes, $\Sigma_{m=1}^{n} x_m$ = constant, are all convex or empty. Then X is symmetry-convex.

Proof. By (2.3.7), it is enough to show that a symmetric set is symmetry-convex if its intersections with the hyperplanes, Σx_m = constant, are all convex or empty. Let Y be such a set, $y \in Y$, and $\sigma \in \Sigma_n$. By the symmetry of Y, $\sigma y \in Y$. Writing $c_o = \Sigma y_m$, it is plain that both y and $\sigma y \mid \cap \{x' x \in R^n; \Sigma x_m = c_o\}$. By the convexity of this set it must therefore contain $[y, \sigma y]$; whence $[y, \sigma y] \subset Y$, as required to prove Y is symmetry-convex and to complete the proof of the theorem.

3. FURTHER CHARACTERIZATIONS OF SYMMETRY-CONVEX SETS

The following theorem links the concept of symmetry-convexity to Problem (1.1).

(3.1) THEOREM. For $X \subset R^n$, the following conditions are equivalent:

(3.1.1) X is symmetry-convex.

(3.1.2) If u and $v \in R^n$, and $\tau \in \Sigma_n$ exchanges i and j such that $u_i \le u_j$ and $v_i < v_j$; then $\tau v - u \in X$ implies $v - u \in X$.

(3.1.3) X is symmetric; and if u and $v \in R^n$, $u_1 \le \dots \le u_n$, $v_1 \le \dots \le v_n$, and $\pi \in \Pi_n$; then $\pi v - u \in X$ implies $v - u \in X$.

(3.1.4) If u and $v \in R^n$, π and $\pi^* \in \Pi_n$, and $(\pi^* v)_i \le (\pi^* v)_j$ whenever $u_i < u_j$; then $\pi v - u \in X$ implies $\pi^* v - u \in X$.

(3.1.5) If u and $v \in R^n$, $v_i \le v_j$ whenever $u_i < u_j$, and $\sigma \in \Sigma_n$; then $\sigma v - u \in X$ implies $v - u \in X$.

Preliminary interpretation. Consider a scheduling situation identical to that of Problem (1.1); but in which a schedule, π, is deemed "acceptable" if and only if $\pi y - x \in X$, where $X \subset R^n$ is a given set. The equivalence of (3.1.1) and (3.1.4) implies that if any schedule is acceptable, then the first-come-first-serve schedules must be acceptable; provided X is a symmetry-convex set. Conversely, if the existence of an acceptable schedule implies that the first-come-first-serve schedules are acceptable, whatever x and y may be; then X must be symmetry-convex.

If "acceptability" is identified with "feasibility," as defined

in Problem (1.1); then, since it obviously follows from (2.4) that the non-negative orthant of R^n is symmetry-convex, the forward implication just made validates the following assertion.

(3.2) ASSERTION. <u>In Problem (1.1), if feasible schedules exist, then the first-come-first-serve schedules are feasible.</u>

The implications, however, go much further. Suppose a schedule, π, is deemed acceptable if and only if its delay cost satisfies $\gamma(\pi y - x) \leq c$, where c is a specified constant. The theorem implies that the first-come-first-serve schedules will meet this criterion whenever any schedule does, provided the set $\{x | \gamma(x) < c\}$ is symmetry-convex; and, conversely, if this is true for every choice of x and y, then $\{x | \gamma(x) \leq c\}$ must be a symmetry-convex set. We shall develop this line of reasoning fully in Section 4.

<u>Proof of (3.1).</u> Suppose (3.1.1) holds, and let u, v, and σ be given as in (3.1.2). Let $\theta = 1$ if $u_i = u_j$; otherwise let $\theta = (u_j - u_i)/(u_j - u_i + v_j - v_i)$. Then it is easily verified algebraically that $0 \leq \theta \leq 1$ and $v - u = \theta(\sigma v - u) + (1 - \theta)\sigma(\sigma v - u)$; whence $v - u \in [\sigma v - u, \sigma(\sigma v - u)]$. Hence, by (2.1) $\sigma v - u \in X$ implies $v - u \in X$, as required to prove (3.1.2).

Suppose (3.1.2) holds, and let u, v, and π be given as in (3.1.3). Let $y^0 = \pi v$, and get y^k from y^{k-1} by exchanging components i and j; where y_i^{k-1} is the least component of y^{k-1} such that $y_i^{k-1} < y_m^{k-1}$ for any $m < i$, and j is the least m for which this inequality holds. It is easily seen that for some K, $y^K = v$; and also that (3.1.2) implies that $y^{k-1} - u \in X$ implies $y^k - u \in X$. It follows

by induction that if $\pi v - u = y^0 - u \in X$, then $v - u = y^K - u \in X$. To prove (3.1.3) it remains to show that X is symmetric; which will follow (as in the proof of (2.3.5)) if it is shown that for $x \in X$ and $\sigma \in \Sigma_n$, $\sigma x \in X$. This is immediate from (3.1.2) with $u = 0$ and $v = \sigma x$.

Suppose (3.1.3) holds, and let u, v, π, and π^* be given as in (3.1.4). It is easily seen that $\rho \in \Pi_n$ can be chosen so that ρu and $\rho\pi^* v$ satisfy the conditions imposed on u and v in (3.1.3). By symmetry, if $\pi v - u \in X$, then $\rho(\pi v - u) \in X$. But

$$\rho(\pi v - u) = (\rho\pi\pi^{*-1}\rho^{-1})(\rho\pi^* v) - (\rho u);$$

whence (3.1.3) implies $\rho\pi^* v - \rho u = \rho(\pi^* v - u) \in X$. Using symmetry again, it follows that $\pi^* v - u \in X$, as required to prove (3.1.4).

Condition (3.1.5) is simply the special case of (3.1.4) in which the identity permutation satisfies the conditions imposed in (3.1.4) on π^*, and is the only π^* considered; and in which the permutations, π, of (3.1.4) are restricted to the elements of Σ_n. Thus, (3.1.4) implies (3.1.5).

Suppose (3.1.5) holds, and let $x \in X$ and $\sigma \in \Sigma_n$. To prove (3.1.1), it is enough to show that if $y \in [x, \sigma x]$, then $y \in X$. Let $y = \theta x + (1 - \theta)\sigma x$, where $0 \leq \theta \leq 1$. Suppose σ exchanges i and j, choosing the notation so $x_i \leq x_j$. Define u and $v \in R^n$ by $u_i = 0$; $u_j = -a$; $u_m = b$ for $m \neq i, j$; $v_i = x_j - a$; $v_j = x_i$; and $v_m = x_m + b$ for $m \neq i, j$; where $a = \theta(x_j - x_i)$, and $b = x_j - \min\{x_m\}$. It is easily verified that $\sigma v - u = x$; whence, by hypothesis, $\sigma v - u \in X$. It is also easily verified that u and v satisfy the condition of (3.1.5), whence $v - u \in X$. Finally, a simple calculation reveals that $v - u = y$; so it follows that $y \in X$, as required.

We have shown that (3.1.1) implies (3.1.2), (3.1.2) implies (3.1.3), (3.1.3) implies (3.1.4), (3.1.4) implies (3.1.5) and (3.1.5) implies (3.1.1), which completes the proof of equivalence.

4. SYMMETRY-CONVEX FUNCTIONS; THE SCHEDULING PROBLEM

The following definition (which relates symmetry-convex functions to symmetry-convex sets in the same way that quasi-convex functions can easily be shown to be related to convex sets) is reconciled with the intention stated in Section 1 by Theorem (4.3) and Assertion (4.4).

(4.1) DEFINITION. Function $\gamma: X \to R$, where $X \subset R^n$, is <u>symmetry-convex</u> if X and all of the sets $\{x|x \in X; \gamma(x) \leq \text{constant}\}$ are symmetry-convex sets.

Our first theorem about symmetry-convex functions states the precise sense in which they are "somewhat convex".

(4.2) THEOREM. <u>Function</u> $\gamma: X \to R$, <u>where</u> $X \subset R^n$, is <u>symmetry-convex if and only if, for</u> $x \in X$, $\sigma \in \Sigma_n$, <u>and</u> $y \in [x, \sigma x]$, <u>we have</u> $y \in X$ <u>and</u> $\gamma(y) \leq \gamma(x)$.

<u>Proof.</u> Let γ by symmetry-convex, and let x, σ, and y be given as in the theorem. That $y \in X$ follows from the requirement that X be symmetry-convex. Let $c = \gamma(x)$. Then, by (4.1) and (2.1), $y \in \{z|z \in X; \gamma(z) \leq c\}$; so $\gamma(y) \leq c$; whence $\gamma(y) \leq \gamma(x)$; and the necessity is proved.

Let γ satisfy the condition of the theorem. By (2.1), X is symmetry-convex. Suppose $x \in \{z|z \in X; \gamma(z) \leq c\}$, $\sigma \in \Sigma_n$, and $y \in [x, \sigma x]$. By the condition of the theorem, $\gamma(y) \leq \gamma(x)$, whence $\gamma(y) \leq c$, and $y \in \{z|z \in X; \gamma(z) \leq c\}$; which proves that this set is symmetry convex, and hence that γ is a symmetry-convex function.

The following theorem is in essence a translation of (3.1) into the language of functions. The routine proofs, based upon (3.1), and each similar to the proof above of (4.2), are omitted.

(4.3) THEOREM. For function $\gamma : X \to R$, where $X \subset R^n$, the following conditions are equivalent:

(4.3.1) γ is symmetry-convex.

(4.3.2) If u and $v \in R^n$, and $\sigma \in \Sigma_n$ exchanges i and j such that $u_i \leq u_j$ and $v_i \leq v_j$; then $\sigma v - u \in X$ implies $v - u \in X$ and $\gamma(v - u) \leq \gamma(\sigma v - u)$.

(4.3.3) γ is a symmetric function (and X a symmetric set); and if u and $v \in R^n$, $u_1 \leq \ldots \leq u_n$, $v_1 \leq \ldots \leq v_n$, and $\pi \in \Pi_n$; then $\pi v - u \in X$ implies $v - u \in X$ and $\gamma(v - u) \leq \gamma(\pi v - u)$.

(4.3.4) If u and $v \in R^n$, π and $\pi^* \in \Pi_n$, and $(\pi^* v)_i \leq (\pi^* v)_j$ whenever $u_i < u_j$; then $\pi v - u \in X$ implies $\pi^* v - u \in X$ and $\gamma(\pi^* v - u) \leq \gamma(\pi v - u)$.

(4.3.5) If u and $v \in R^n$, $v_i \leq v_j$ whenever $u_i < u_j$, and $\sigma \in \Sigma_n$; then $\sigma v - u \in X$ implies $v - u \in X$ and $\gamma(v - u) \leq \gamma(\sigma v - u)$.

Interpretation. Suppose again that in Problem (1.1) a schedule, π, is deemed "acceptable" if and only if $\pi y - x \in X$, where $X \subset R^n$ is a given set; and interpret this set X as the domain of the delay-cost function, γ. The equivalence of (4.1.1) and (4.1.4) implies that if acceptable schedules exist, then the first-come-first-serve schedule will minimize delay cost over the acceptable schedules; provided γ is a symmetry-convex function. Conversely, if the existence of an acceptable schedule implies that the first-come-first-serve schedule will minimize delay cost, then γ must be a symmetry-convex function.

Specialized to the case in which "acceptibility" and "feasibility" are identified, this yields the answer to the main question raised in Section 1.

(4.4) ASSERTION. <u>In problem (1.1), the first-come-first-serve schedules minimize delay cost whenever feasible schedules exist, if and only if the delay-cost function is symmetry-convex.</u>

(The domain of the delay-cost function in Problem (1.1) must, of course, be the non-negative orthant of R^n.)

Equivalent condition (4.3.3) emphasizes that the different jobs must play symmetric roles with respect to delay cost if we are to be sure that the first-come-first-serve schedules will minimize this cost, and interprets these schedules simply: we "may as well" number the jobs consecutively in order of arrival, number the processing epochs consecutively, and assign each processing epoch to the identically numbered job. Equivalent condition (4.3.5) points out that the conditions on γ are not actually weakened if we merely require that no first-come-first-serve schedule be subject to improvement (i.e., reduction in delay cost) by exchanging the processing epochs of a single pair of jobs. This condition can be reformulated parallel to (4.3.3); or, more simply, (4.3.3) remains an equivalent condition to the others if "$\pi \in \Pi_n$" and "π" are replaced throughout by "$\tau \in \Sigma_n$" and "τ".

Condition (4.3.2) is of particular interest because it facilitates inductive proofs of theorems relating to more complex situations than those to which this paper is devoted. Suppose in particular that the x_m are reinterpreted as "due-dates," and that new variables, say z_m, are introduced as "arrival epochs"; where the z_m now determine which schedules are feasible, as do the x_m in Problem (1.1), but the x_m pertain to "tardiness cost" precisely as they do to delay cost in Problem (1.1).

It can be shown, using (4.3.2), that the symmetry-convex tardiness-cost functions are those for which "first-due-first-serve" schedules, defined in a natural way, minimize tardiness cost whenever feasible schedules exist. The rigorous formulation and proof of this result is left for another paper, in which its implications for a wider range of "due-date scheduling problems" will also be developed.

<u>A special case.</u> Because the symmetry-convex functions are unfamiliar, it seems appropriate to conclude this section with special cases of the sufficiency in (4.4), concerned with functions defined using classical concepts; and to give some examples. The following two lemmas, of some interest in themselves, directly imply Theorem (4.7), whose sufficient condition that a function be symmetry-convex is incorporated into Assertion (4.9), below.

(4.5) LEMMA. <u>If $\gamma: X \to R$, where $X \subset R^n$ is a symmetry-convex set, is an infimum of symmetry-convex functions, then γ is symmetry-convex.</u>

<u>Proof.</u> Let $\gamma = \inf_{a \in A} \{\gamma_a\}$, where A is arbitrary. Suppose $x \in X$, $\sigma \in \Sigma_n$, and $y \in [x, \sigma x]$. For $\epsilon > 0$ there exists $a \in A$ such that $\gamma(x) > \gamma_a(x) - \epsilon$. By (4.2), since γ_a is symmetry-convex, $\gamma_a(x) \geq \gamma_a(y)$; whence it follows that $\gamma(x) > \gamma(y) - \epsilon$. Since ϵ is arbitrary, it follows that $\gamma(x) \geq \gamma(y)$; whence the conclusion follows from (4.2).

(4.6) LEMMA. <u>A quasi-convex symmetric function is symmetry-convex.</u>

<u>Proof.</u> Let $\gamma: X \to R$ be a quasi-convex symmetric function. From the definition (Section 1), X is convex and symmetric; whence, by (2.4), X is symmetry-convex; and the first part of the condition of Theorem (4.2) is satisfied.

Further, if $x \in X$, $\sigma \in \Sigma_n$, and $y \in [x,\sigma x]$; then $\gamma(x) = \gamma(\sigma x)$, by symmetry; and, by quasi-convexity, $\gamma(y) \leq \max\{\gamma(x); \gamma(\sigma x)\} = \gamma(x)$; which verifies the second part of the condition of (4.2), showing that γ is symmetry-convex, and completing the proof.

(4.7) THEOREM. Let function $\gamma: X \to R$, where $X \subset R^n$ is a symmetry-convex set, be an infimum of quasi-convex symmetric functions. Then γ is symmetry-convex.

We omit the routine proof of the following theorem, whose condition is also incorporated into (4.9), but draw attention to the significantly changed position of the word, "symmetric," as compared to (4.7).

(4.8) THEOREM. Let function $\gamma: X \to R$, where $X \subset R^n$ is a symmetry-convex set, be a symmetric supremum of quasi-convex functions. Then γ is symmetry-convex (in fact, quasi-convex, if X is convex).

Our next assertion is immediate from (4.4), (4.7), and (4.8).

(4.9) ASSERTION. In Problem (1.1), suppose the delay-cost function is either an infimum of quasi-convex symmetric functions, or a symmetric supremum of quasi-convex functions. Then, whenever feasible schedules exist, the first-come-first-serve schedules minimize delay cost.

Examples. If $\gamma(x) = \max\{g(x_m)\}$ over $m = 1,\ldots,n$, where $g: R \to R$ is quasi-convex; then γ is symmetry-convex, by (4.8). In particular, $\max\{x_1;\ldots;x_n\}$ is symmetry-convex; and so is the function defined as the number of x_m which exceed any specified bound.

Thus, in Problem (1.1), the first-come-first-serve schedules minimize the maximum delay, and also the number of jobs subject to delay exceeding any specified duration.

If $\gamma(x) = \Sigma_{m=1}^{n} g(x_m)$, where $g:R \to R$ is convex; then it is easily verified directly that γ is convex (and hence quasi-convex) and symmetric; whence by a "degenerate" application of either (4.7) or (4.8), γ is symmetry-convex. Thus, in Problem (1.1), if the delay cost is the sum of costs attached to the individual jobs, each determined by the same convex function of the jobs' waiting times; then the delay cost is minimized by the first-come-first-serve schedules. Plausible specific functions in this category include the sum of the waiting times, and the sum of the squares of the waiting times.

5. CONCLUSION

One can easily construct symmetric functions which are not symmetry-convex. The appearance of such a delay-cost function in Problem (1.1) not only dissolves our assurance that the first-come-first-schedules must minimize delay cost; but in fact guarantees that the first-come-first-serve schedules will not be optimal, for at least some sequences of arrival and processing epochs. The same is true if a non-symmetric delay-cost function appears.

But, within the realm of symmetric functions, it is difficult to avoid artificiality and symmetry-convexity at the same time. In fact, the functions covered by Assertion (4.9) seem to comprehend the full range of plausible delay-cost functions which are symmetric.

We conclude that, in practical situations like that of Problem (1.1), it is generally safe to assume that the first-come-first-serve schedules will minimize delay cost whenever feasible schedules exist, provided the delay cost is a symmetric function of the waiting times.

As was remarked above, this dictum remains valid for a wide range of more complex and realistic scheduling problems; and these problems will be explored in another paper.

APPENDIX

FURTHER CHARACTERIZATIONS OF SYMMETRY-CONVEX SETS AND FUNCTIONS

Although the practical significance of the more complex symmetry-convex sets is doubtful, along with that of symmetry-convex functions beyond those described by Theorems (4.7) and (4.8) of the text; it is of some interest (and potential practical importance with respect to presently unanticipated applications) to have geometric characterizations of the sets, and complete characterizations of the functions based on classical convexity concepts. This appendix provides such characterizations.

A key lemma. The results of this appendix are based upon the following lemma.

(A.1) LEMMA. For $x \in R^n$, let $C(x)$ be the convex hull of the set, $\{\pi x \mid \pi \in \Pi_n\}$. Then $C(x) = S(x)$.

Proof. Since $C(x)$ is obviously symmetric, and convex by definition, (2.4) implies that $C(x)$ is symmetry-convex. Since $x \in C(x)$, it follows by (2.3.6) and (2.3.3) that $C(x) \supset S(x)$.

It remains to show that $C(x) \subset S(x)$. Suppose it proved that, if $y_1 \leq \ldots \leq y_n$, then $C(y) \subset S(y)$. Given arbitrary $x \in R^n$, it is easily seen that we can choose $\pi \in \Pi_n$ such that $(\pi x)_1 \leq \ldots \leq (\pi x)_n$. Since $C(x)$ is obviously symmetric, and $S(x)$ is symmetric by (2.3.4) and (2.3.5), we have $C(x) = C(\pi x)$ and $S(x) = S(\pi x)$. By supposition, $C(\pi x) \subset S(\pi x)$; so it follows that $C(x) \subset S(x)$.

It remains to show that, if $x_1 \leq \ldots \leq x_n$, then $C(x) \subset S(x)$. We do so by induction.

Let $C_M(x)$ be the set of $y \in R^n$ which are expressible in the form, $y = \Sigma\, \theta_\pi\, \pi x$, where $0 \leq \theta_\pi \leq 1$ and $\Sigma\, \theta_\pi = 1$, and where $\theta_\pi = 0$ unless $\pi(m) = m$ for $m > M$. For $M = 0$, the one permutation which satisfies this last condition is the identity, whence $C_0(x) = \{x\}$. Hence, by (2.3.1), $C_0(x) \subset S(x)$. For M sufficiently large ($M = n$ is obviously large enough), the restriction on permutations is vacuous; whence $C_M(x) = C(x)$.

To complete a proof by induction, it remains to show that, for $1 \leq M$, if $x_1 \leq \ldots \leq x_n$, then $C_{M-1}(x) \subset S(x)$ implies $C_M(x) \subset S(x)$. Suppose the same statement proved, but with $C_M(x)$ replaced by

$$D_M(x) = \{y \mid y \in C_M(x);\ y_M = \max\,\{y_m\} \text{ over } m \leq M\}.$$

Let $y \in C_M(x)$. It is plain that we can choose $\pi \in \Pi_n$ such that $(\pi y)_M = \max\,\{y_m\}$ over $m \leq M$, and also $\pi(m) = m$ for $m > M$. It is easily verified that $\pi y \in D_M(x)$; whence, by supposition, $\pi\, y \in S(x)$. Consequently, by the symmetry of $S(x)$, $y \in S(x)$. It follows that $C_M(x) \subset S(x)$.

It remains to show that, for $1 \leq M$, if $x_1 \leq \ldots \leq x_n$, then $C_{M-1}(x) \subset S(x)$ implies $D_M(x) \subset S(x)$. We do so by induction. Let $D_{M,N}(x)$ be the set of $y \in D_M(x)$ which are expressible in the form, $y = \Sigma\, \theta_\pi \pi x$, in accord with the conditions imposed on similar expressions representing elements of $C_M(x)$, and with the added restriction that there be at most N different permutations π such that $\theta_\pi > 0$ and $\pi(M) \neq M$. If $N = 0$, then $\theta_\pi = 0$ unless $\pi(M) = M$, whence in fact $\theta_\pi = 0$ unless $\pi(m) = m$ for $m > M - 1$; and it follows that $D_{M,0}(x) \subset C_{M-1}(x)$. Consequently, by supposition, $D_{M,0}(x) \subset S(x)$. For N sufficiently large ($N = n!$ is obviously large enough), the restriction delineating $D_{M,N}(x)$ as a subset of $D_M(x)$ is vacuous; whence $D_{M,N}(x) = D_M(x)$.

To complete a proof by induction, it remains to show that, for $1 \leq M$ and $1 \leq N$, if $x_1 \leq \ldots \leq x_n$, then $D_{M,N-1}(x) \subset S(x)$ implies $D_{M,N}(x) \subset S(x)$; i.e., $D_{M,N-1}(x) \subset S(x)$ implies that if $y \in D_{M,N}(x)$ then $y \in S(x)$.

Let $y \in D_{M,N}(x)$. If $y \in D_{M,N-1}(x)$; then, by supposition, $y \in S(x)$, and we are done. Otherwise, we can write $y = \Sigma\, \theta_\pi\, \pi x$, where $0 \leq \theta_\pi \leq 1$ and $\Sigma\, \theta_\pi = 1$, where $\theta_\pi = 0$ unless $\pi(m) = m$ for $m > M$, and where $\theta_\pi > 0$ for exactly N permutations π such that $\pi(M) \neq M$. Since $N \geq 1$, we can choose $\rho \in \Pi_n$ such that $\theta_\rho > 0$ and $\rho(M) \neq M$. Let $i = \rho(M)$ and $j = \rho^{-1}(M)$. Let $\sigma \in \Sigma_n$ exchange i and M. Let $z = y + \theta_\rho\, (\sigma\rho x - \rho x)$.

From the above representation for y, a representation, $z = \Sigma\, \varphi_\pi\, \pi x$, of z can be derived which is identical to that for y except that ρ has been "eliminated" and $\sigma\rho$ has been "introduced" or its coefficient has been increased. Also, $(\sigma\rho)(j) = \sigma(\rho(j)) = i$; and similarly $(\sigma\rho)(M) = M$ and $(\sigma\rho)(m) = \rho(m)$ for $m \neq j, M$. Since obviously $j < M$, it follows that this representation of z satisfies the conditions imposed on the above representation of y, but with N replaced by $N - 1$. By straightforward calculations (remembering that $(\pi x)_m = x_{\pi^{-1}(m)}$), we find that $z_i = y_i - \theta_\rho\, (x_M - x_j)$, $z_M = y_M + \theta_\rho\, (x_M - x_j)$, and $z_m = y_m$ for $m \neq i, M$. It follows, using the assumption that $x_1 \leq \ldots \leq x_n$, that $z_i \leq y_i$ and $z_M \geq y_M$. Since $y \in D_M(x)$, we have $y_M = \max\, \{y_m\}$ over $m \leq M$; which, with the relationships just established between components of y and z, implies that $z_M = \max\, \{z_m\}$ over $m \leq M$. This, with the previous conclusions about the representation, $z = \Sigma\, \varphi_\pi\, \pi\, x$, implies that $z \in D_{M,N-1}(x)$. Hence, by supposition, $z \in S(x)$. Consequently, by (2.3.4) and (2.1), $[z, \sigma z] \subset S(x)$.

To show that $y \in S(x)$, and thus complete the proof of the lemma, we need only show that $y \in [z, \sigma z]$; i.e., that $y = \theta z + (1 - \theta)\sigma z$, for some θ such that $0 \leq \theta \leq 1$. It is easy to verify algebraically that the required conditions are satisfied by $\theta = 1$ if $z_i = z_M$, and otherwise by $\theta = (z_M - y_i) / (z_M - z_i)$.

Characterization of symmetry-convex sets. By (A.1) and (2.3.9), set $X \subset R^n$ is symmetry-convex if and only if $X = \cup_{x \in X} C(x)$. The $C(x)$ are obviously symmetric sets whose intersections with the hyperplanes, $\Sigma_{m=1}^{n} x_m$ = constant, are all convex or empty; so the following theorem is an immediate consequence of the statement just made and Theorem (2.4).

(A.2) THEOREM. For $X \subset R^n$, the following conditions are equivalent:

(A.2.1) X is symmetry-convex.

(A.2.2) $X = \cup_{x \in X} C(x)$, where $C(x)$ is the convex hull of the set $\{\pi x \mid \pi \in \Pi_n\}$.

(A.2.3) X is a union of symmetric sets whose intersections with the hyperplanes, $\Sigma_{m=1}^{n} x_m$ = constant, are convex or empty.

It is clear that the symmetry-convexity of a set $X \subset R^n$ is actually a property of its intersections with the hyperplanes, $R_c^n = \{x \mid x \in R^n;\ \Sigma_{m=1}^{n} x_m = c\}$, for $c \in R$; i.e., every symmetry-convex set is a union of (disjoint) symmetry-convex sets contained in the hyperplanes R_c^m, and every such union is a symmetry-convex set. Thus, the structure of the symmetry-convex sets in R^n is completely describable in terms of the structure of the symmetry-convex subsets of R_c^n.

It is easily verified that the symmetry-convex subsets of R_c^2 are identical with the convex symmetric subsets of R_c^2; but the situation becomes more complicated in R^3 and in higher-dimensional spaces.

In R^3, consider, for instance, $X = C(1,1,4) \cup C(0,3,3)$. Each of the two sets conjoined is an equilateral triangle (with its interior) in the plane $x_1 + x_2 + x_3 = 6$. Further, these triangles are concentric, and one is rotated 60° with respect to the other. Thus, X is a Star of David (with its interior). Similarly, $C(0.9,0.9,3.8) \cup C(0.2,2.9,2.9)$ is a Star of David with its points clipped short (but not equally short). The reader who wants to get a "feel" for the symmetry-convex sets will do well to examine those which are subsets of R_c^3, starting with the $C(x)$ and with the unions of small collections of these sets.

Characterization of symmetry-convex functions. It is easily seen from the remarks just made and Definition (4.1) that there are symmetry-convex functions whose isoclines have the same shape as the outer boundary of a Star of David; which means that the "cross-section" of such a function along a segment like $[(0,3,3), (1,1,4)]$ can rise higher than the (possibly equal) functional values at the endpoints. Thus, the symmetry-convex functions are not all quasi-convex. The exercise suggested above for exploring the symmetry-convex sets, if carried out with Definition (4.1) in mind, will also help to get a "feel" for the symmetry-convex functions.

As in the case of symmetry-convex sets, it is easily seen that a symmetry-convex function need be no more than a conglomerate of functions defined on subsets of the R_c^n, unrelated to one another except by name. We thus suffer no loss by concentrating on symmetry-convex functions with domains contained in the R_c^n.

First, let $\gamma: X \to R$ be a bounded symmetry-convex function, where $X \subset R_c^n$; and choose $M \geq \text{lub}\,\{\gamma\}$. Let $A = \{(d,z) | d \in R;\ z \in X; \gamma(z) \leq d\}$. For $a = (d,z) \in A$, define $\gamma_a: X \to R$ by $\gamma_a(x) = d$ if $x \in C(z)$, and $\gamma(x) = M$ if $x \notin C(z)$; where $C(z)$ is defined as in (A.1) and (a.2.2).

It is easily verified that the γ_a are quasi-convex and symmetric, and that $\gamma = \inf_{a \in A} \{\gamma_a\}$. With (4.7) this yields the following theorem.

(A.3). Theorem. <u>Suppose function</u> $\gamma: X \to R$ <u>is bounded, where</u> $X \subset R_c^n$. <u>Then</u> γ <u>is symmetry-convex if and only if it is an infimum of quasi-convex symmetric functions.</u>

The theorem in fact holds, and the above proof is valid, if R_c^n in its statement is replaced by R^n. The restriction to bounded functions is then more severe.

To obtain a complete characterization, we define a natural extension of the concept of infima of collections of functions. Function $\gamma: X \to R$ is the exinfimum of the collection, $\{\gamma_a | a \in A\}$, of functions $\gamma_a: X_a \to R$, provided $X \subset \bigcup_{a \in A} X_a$, and for each $x \in X$, $\gamma(x) = \inf \{\gamma_a(x)\}$ where the "inf" is taken over as such that $x \in X_a$. We then write $\gamma = {}_x\inf \{\gamma_a\}$.

Now, using the notations of the paragraph preceding (A.3), for $a = (d,z) \in A$, define $\gamma_a: C(z) \to R$ by $\gamma_a(x) \equiv d$. It is easily verified that the γ_a are convex and symmetric, and that $\gamma = x\inf \{\gamma_a\}$. The proof of (4.7) actually applies to exinfima (and the convex functions are quasi-convex); so the result just obtained, with (4.7), yields the next theorem.

(A.4). THEOREM. <u>Function</u> $\gamma: X \to R$, <u>where</u> $X \subset R_c^n$, <u>is symmetry-convex if and only if it is an exinfimum of convex symmetric functions</u>.

Theorem (A.4) holds without other modification, and the proof given remains valid, if R_c^n is replaced in the statement by R^n. But it is easily seen, from the nature of "exinfima" that nothing is really added by this modification.

An alternative innovation to the invention of the exinfima is to define "exconvex" functions in the same way as ordinary convex functions, except permitting positive-infinity as a functional value. Theorem (A.4) is valid if "exinfimum" is replaced by "infimum" and "convex" by "exconvex." (The statement got by making the first of these changes but not the second is false.)

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This paper develops the theory of "symmetry-convex" functions, whose appearance in certain scheduling problems as delay-cost functions (of waiting times) guarantees that the "first-come-first-serve" schedules will be optimal. These are the symmetric functions satisfying a mild convexity condition (which appears to hold in most if not practical scheduling situations). The conclusion of the present paper is that, for a rather special type of scheduling problem, the first-come-first-serve schedules are optimal, provided the waiting times of individual jobs affect delay cost symmetrically. Extensions of this conclusion (to be detailed in one or more subsequent papers) cover broad classes of scheduling problems of great practical significance, relating to both "first-come-first-serve" and "first-due-first-serve" scheduling.

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